Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 1, pp. 105–112, 1996

0021–8928/96 $24.00+0.00

S0021–8928(96)00014-7

THE METHOD OF HOMOGENEOUS SOLUTIONS AND BIORTHOGONAL EXPANSIONS IN THE PLANE PROBLEM OF THE THEORY OF ELASTICITY FOR AN ORTHOTROPIC BODY†

V. V. VASIL'YEV and S. A. LUR'YE

Moscow

(*Received* 28 *July* 1994)

In connection with the solution of boundary-value problems of the theory of elasticity for an orthotropic strip the problem of expanding two different limiting functions in series in terms of the characteristic elements of the generalized eigenvalue problem is considered. A system of functions biorthogonal to the system of characteristic elements is constructed. The double completeness of characteristic elements is proved. It is shown that the biorthogonality condition is equivalent to a generalized orthogonality relation of Papkovich type. The form of systems of biorthogonal functions is established. For expansions of a special form the biorthogonal systems are identical with the systems of characteristic elements. Biorthogonal systems of functions are constructed corresponding to expansions of a general form. Using the biorthogonal systems obtained, explicit expressions for the expansion coefficients are found. An example demonstrating the existence of a non-trivial double null expansion is given.

1. CHARACTERISTIC ELEMENTS AND DOUBLE EXPANSIONS

We consider the plane problem of the theory of elasticity for a rectangular orthotropic strip in dimensionless coordinates x and y ($|x| \leqslant 1$, $|y| \leqslant 1$). The strip has a half-length a and a half-width b in x and y coordinates. Homogeneous boundary conditions are given on the longitudinal edges $y = \pm 1$ of the strip.

The solution of the resolving equation of the problem can be constructed as an expansion in terms of homogeneous solutions corresponding to the generalized biharmonic problem [1]. Henceforth we shall assume for simplicity that the problem is symmetric in the central coordinates x and y. Using the method of homogeneous solutions, we write

$$\varphi(x,y) = \sum_{n=1}^{\infty} A_n \operatorname{ch} \lambda_n x F_n(y) = \varphi_n(x,y) \tag{1.1}$$

for the stress function, where $\varphi_n(x, y)$ is a partial solution, and $F_n(y)$ and λ_n are the eigenfunctions and eigenvalues of the following eigenvalue problem [1]

$$F_n'''' + 2p\lambda_n^2 F_n'' + q\lambda_n^4 F_n = 0 \tag{1.2}$$

$$y = \pm 1: \quad F_n'' + \beta\lambda_n^2 F_n = 0, \quad F_n''' + \alpha\lambda_n^2 F_n' = 0 \tag{1.3}$$

Here p, q, α, β are real constants.

The eigenfunctions have the form

$$F_n(y) = u_{1n}(y) + u_{2n}(y) \tag{1.4}$$
$$u_{in}(y) = C_{in} \cos t_i \lambda_n y, \quad C_{1n} = a_2 \cos t_2 \lambda_n, \quad C_{2n} = -a_1 \cos t_1 \lambda_n$$

The eigenvalues λ_n are the roots of the characteristic equation

$$\Psi(\lambda) = a_1 b_2 t_2 \cos t_1 \lambda \sin t_2 \lambda - a_2 b_1 t_1 \sin t_1 \lambda \cos t_2 \lambda = 0 \tag{1.5}$$
$$a_i = t_i^2 - \beta, \quad b_i = t_i^2 - \alpha$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 1, pp. 111–109, 1996.

We shall call u_{in} the characteristic elements. The eigenfunctions $F_n(y)$ satisfy the general orthogonality relation, which can be expressed as

$$a_{kn} = \int_{-1}^{1} [t_1^2 c_1 u_{1n}(y) u_{1k}(y) - t_2^2 c_2 u_{2n}(y) u_{2k}(y)] dy =$$

$$= \int_{-1}^{1} [D_{1n}(y) C_{1k} \cos t_1 \lambda_k y - D_{2n}(y) C_{2k} \cos t_2 \lambda_k y] dy = 0, \quad \lambda_n \neq \lambda_k \tag{1.6}$$

$$c_i = \beta\alpha + t_i^2(t_i^2 - \beta - \alpha), \qquad D_{in}(y) = t_i^2 c_i C_{in} \cos t_i \lambda_n y$$

The generalized orthogonality relation enables us to obtain an exact solution of the problem concerned with the expansions

$$\varphi_i(y) = \sum_{n=1}^{\infty} A_n \lambda_n^2 u_{in}(y), \quad i = 1,2 \tag{1.7}$$

the coefficients of which can be found from the formulae

$$A_k = \frac{d_k}{\lambda_k^2 a_{kk}}, \qquad d_k = \int_{-1}^{1} [\varphi_1 D_{1k}(y) - \varphi_2 D_{2k}(y)] dy \tag{1.8}$$

Substitution of the stress functions into the boundary conditions at the ends $x = \pm 1$ of the region leads to the expansions

$$\sum_{n=1}^{\infty} A_n [\eta_1(\lambda_n) u_{1n}(y) + \xi_1(\lambda_n) u_{2n}(y)] = f_1(y), \quad (1 \leftrightarrow 2) \tag{1.9}$$

where $f_i(y)$ are known functions and η_i and ξ_i are certain entire even functions of λ_n.

2. COMPLETENESS OF A SYSTEM OF VECTOR-VALUED FUNCTIONS

We shall study the completeness of the system of generalized eigenfunctions. We note that the double completeness of the homogeneous functions $\{F_k(\lambda_k, y)\}_{k=1}^{k=\infty}$, $F_k(1) = F_k'(1) = 0$ was proved in [2] using the theory of quadratic bundles.

Here, to prove completeness, we shall use the possibility of constructing a system of functions biorthogonal to the system under investigation [3], as well as the uniqueness theorem for entire functions [4].

The function $\Psi(\lambda)$ defined by (1.5) is an entire function of exponential type. Its type is equal to $\sigma = t_1 + t_2$.

We shall now consider the vector-valued function $R = (C_1 \cos t_1 \lambda y, C_2 \cos t_2 \lambda y)$. Let $\psi_i(y)$ be square integrable functions in the interval $(-1, 1)$.

We consider the expressions

$$X_{i\theta}(\lambda) = \int_{-\theta}^{\theta} C_i(\lambda) \cos t_i \lambda x \psi_i(x) dx$$

Proposition 1. If $\theta \leqslant 1$, then $X_{1\theta}(\lambda) + X_{2\theta}(\lambda)$ is an entire function of exponential type, its type satisfying the inequality $\sigma \leqslant t_1 + t_2$.

To prove this proposition we set $\lambda = u_1 + iu_2$. Then, by Schwartz's inequality, we get

$$X_{1\theta} \leqslant \alpha_2 \exp[(t_2 + \theta t_1)|\lambda|], \quad X_{2\theta} \leqslant \beta \exp[(t_1 + \theta t_2)|\lambda|]$$

where α_1, α_2 and β are positive constants. The set of these inequalities proves the proposition.

The double completeness will be proved using the well-known scheme [3]. We begin by proving that the vector-valued function $R_a = \{R_{a1}(\lambda, y), R_{a2}(\lambda, y)\}$, $(R_{a1}(\lambda, y) = a_2 C_1 \cos t_1 \lambda y$, $R_{a2}(\lambda, y) = a_1 C_2 \cos t_2 \lambda y)$ forms a closed kernel in $L_2(-1, 1)$. This means that there is no finite vector-valued function $\kappa(y) \in L_2(-1, 1)$, $\kappa = \{\kappa_1(y), \kappa_2(y)\}$ non-equivalent to zero if the equality

$$G_a(\lambda) = \int_{-1}^{1} [R_{a1}(\lambda, y) \kappa_1(y) + R_{a2}(\lambda, y) \kappa_2(y)] dy = 0 \tag{2.1}$$

is satisfied.

Here $R_{a1}(\lambda, y)$ and $R_{a2}(\lambda, y)$ ($\lambda \in C$, supp $R_{ai}(\lambda, y) \in [-1, 1]$, $i = 1, 2$) are functions generating the system $F_a(\lambda_k, y) = R_{a1}(\lambda_k, y) + R_{a2}(\lambda_k, y)$ for $\lambda \in \Psi$. The functions $C_i(\lambda)$ have been defined before.

Proposition 2. The vector-valued function $R_a(\lambda, y)$ is a closed kernel in $L_2(-1, 1)$. In other words, the space of zeros of the functional $G_a(\lambda)$ is closed in $L_2(-1, 1)$.

Proof. Since the functions $C_i(\lambda)$ are such that $F_a(\lambda, 1) = 0$ and $\kappa_i(y)$ are finite functions, the solution of Eq. (2.1) in the space of Fourier transforms has the form

$$F_{t_i\lambda}[\kappa_i(y)] = \cos t_i\lambda$$

Correspondingly, the solution in the space of original functions is given by finite functions vanishing inside the interval $(-1, 1)$, $\kappa_i(y) = d_i[\delta(y + 1) + \delta(y - 1)]$, where d_i are constants and $\delta(\,)$ is the delta-function. It follows that R_α is a closed kernel in $L_2(-1, 1)$.

Theorem 1. The system $\{R_a(\lambda_k, y)\}_{k=1}^{\infty}$ of vector-valued functions is complete in $L_2(-1, 1)$.

Proof. We take the function

$$S_\theta(\lambda) = \int_{-\theta}^{\theta} R_a(\lambda, y)\kappa(y)dy, \quad \lambda \in C, \quad 0 < \theta < 1$$

Here we assume that κ is a vector-valued function whose components $\kappa_i(y)$ are finite functions from $L_2(-\theta, \theta)$, (supp $\kappa_i(y) \in [-\theta, \theta]$, $0 < \theta < 1$).

Let the $\kappa_i(y)$ be such that

$$S_\theta(\lambda_k) = 0, \quad \lambda_k \in \Psi \tag{2.2}$$

According to Proposition 1, $S_\theta(\lambda_k)$ is an entire function whose type does not exceed $\max(t_1 + \theta t_2, t_1\theta + t_2)$. On the other hand, each root of the entire function $\Psi(\lambda)$, whose type is equal to $t_1 + t_2$, is, by assumption, a zero of $S_\theta(\lambda)$. By the uniqueness theorem for entire functions, we find that $S_\theta(\lambda) \equiv 0$ for $\theta < 1$ Proposition 2, $R_a(\lambda, y)$ is a closed kernel in $L_2(-1, 1)$. It follows that the system of vector-valued functions $\{R_a(\lambda_k, y)\}_{k=1}^{\infty}$ is complete in $L_2(-1, 1)$.

Remark. In exactly the same way one can prove the completeness of another system of vector-valued functions. This system, which satisfies the boundary conditions $F''' - n\lambda^2F' = 0$ identically for $y = \pm 1$, has the form

$$R_b = \{R_{b1}(\lambda_k, y),\ R_{b2}(\lambda_k, y)\}_{k=1}^{\infty}$$
$$R_{b1} = b_2D_1\cos t_1\lambda_k y, \quad R_{b2} = b_1D_2\cos t_2\lambda_k y$$
$$D_1 = -b_2t_2\sin t_2\lambda_k, \quad D_2 = -b_1t_1\sin t_1\lambda_k$$
$$C_1 = b_i = t_i^2 + n - b_2t_2\sin t_2\lambda, \quad C_2 = b_1t_1\sin t_1\lambda$$

The completeness of the system of vector-valued functions $\{R_a(\lambda_k y)\}_{k=1}^{\infty}$ and $\{R_b(\lambda_k y)\}_{k=1}^{\infty}$ in $L_2(-1, 1)$ is equivalent to the double completeness of the system $\{R(\lambda_k, y)\}_{k=1}^{\infty}$.

Now we consider the expression

$$G(\lambda) = \int_{-1}^{1}[u_1(\lambda, y)\psi_1(y) + u_2(\lambda, y)\psi_2(y)]dy$$
$$(u_i(\lambda, y) = C_i(\lambda)\cos t_i\lambda y, \quad i = 1, 2)$$

By the definition of $C_1(\lambda)$ and $C_2(\lambda)$ and Proposition 1, $G(\lambda)$ is an entire function of exponential type. Its type is equal to $t_1 + t_2$. We assume $\psi_{1k}(y)$ and $\psi_{2k}(y)$ to be such that

$$G(\lambda) = \int_{-1}^{1} [u_1(\lambda, y)\psi_{1k}(y) + u_2(\lambda, y)\psi_{2k}(y)]dy = \frac{2\lambda\Psi(\lambda)}{(\lambda^2 - \lambda_k^2)\Psi'(\lambda_k)} \tag{2.3}$$

If this system of functions exists, then one can say that a system of vector-valued functions

$\{\psi_k\}_{k=1}^{\infty} = \{\psi_{1k}(y), \psi_{2k}(y)\}_{k=1}^{\infty}$ biorthogonal to $\{R(\lambda_i)\}_{i=1}^{\infty}$, $\lambda \in \Psi$ exists. Indeed, if $i \neq k$, then $G(\lambda_i) = 0$, so that λ_i are included among the zeros of $\Psi(\lambda)$. When $i \neq k$, then $G(\lambda_k) = 1$ by (2.3).

Comparison of the generalized orthogonality condition (1.6) with (2.3) leads to the idea of an alternative treatment of the extended orthogonality relations for homogeneous functions as biorthogonality relations. One only needs to demonstrate that a biorthogonal system of vector-valued functions with the given properties exists.

Theorem 2. There is a system of finite vector-valued functions $\{\psi_k(y)\}_{k=1}^{\infty} = \{\psi_{1k}(y), \psi_{2k}(y)\}_{k=1}^{\infty}$ with supports coinciding with the interval $[-1, 1]$ that are biorthogonal to the system of vector-valued functions $\{R(\lambda_k, y)\}_{k=1}^{\infty}$, $\lambda \in \Psi$ generated by the eigenfunctions of problem (1.2), (1.3).

Proof. We consider a function $G(\lambda)$. Assuming that $\psi_{1k}(y)$, $\psi_{2k}(y)$ are finite functions, we write it as

$$G(\lambda) = \int_{-\infty}^{\infty} [u_1(\lambda, y)\psi_{1k}(y) + u_2(\lambda, y)\psi_{2k}(y)]dy =$$

$$= C_1(\lambda)F_{t_1\lambda}[\psi_{1k}(y)] + C_2(\lambda)F_{t_2\lambda}[\psi_{2k}(y)] = C_1(\lambda)X_{1k}(\lambda) + C_2(\lambda)X_{2k}(\lambda) \tag{2.4}$$

Here $F_{t_i\lambda}[\psi_{ik}(y)]$ is the integral transformation of $\psi_{ik}(y)$ with transformation parameter $r_i = t_i\lambda$, $F_{r_i}[\psi_{ik}(y)] = \mathbf{X}_{ik}(r_i)$ $(i = 1, 2)$. The question of the existence of finite functions on $(-1, 1)$ that are biorthogonal to the system has an affirmative answer if functions exists having the following properties:

1. $\mathbf{X}_{ik}(r_i)$ must be entire functions of exponential type, their type being equal to 1, $r_i = t_i\lambda$;
2. $\mathbf{X}_{ik}(r_i)$ satisfy the equality

$$C_1(\lambda)X_{1k}(t_1\lambda) + C_2(\lambda)X_{2k}(t_2\lambda) = \frac{2\lambda\Psi(\lambda)}{(\lambda^2 - \lambda_k^2)\Psi'(\lambda_k)} \tag{2.5}$$

Such functions exist and have the form

$$X_{ik}(t_i\lambda) = a_{ik}\frac{\cos t_i\lambda_k[R_{al}(\lambda_k, 1)R_{bi}(\lambda, 1) - R_{ai}(\lambda, 1)R_{bl}(\lambda_k, 1)]}{a_l b_i \cos(t_i\lambda_k)[(t_i\lambda)^2 - (t_i\lambda_k)^2]} \tag{2.6}$$

$$i = 1, 2, \qquad l = 1, 2, \qquad t \neq l, \qquad \lambda_k \in \Psi$$

Here a_{1k} and a_{2k} are certain constants, which must be determined.

It is obvious that $\mathbf{X}_{ik}(r_i)$, $r_i = t_i\lambda$ $(i = 1, 2)$ are entire functions of exponential type, their type being equal to one. By the Paley–Wiener theorem [5], $\psi_{ik}(y)$ are finite functions with support $\psi_i(y) \in [-1, 1]$ $(i = 1, 2)$.

We shall show that condition (2.5) is satisfied for a suitable choice of the coefficients a_{ik} in (2.6). Indeed, one can first verify that the expressions with the coefficients a_{1k} and a_{2k} on the right-hand sides of (2.6) are equal, respectively, to the integrals

$$\int_{-1}^{1} \cos t_i\lambda y \cos t_i\lambda_k y\, dy, \qquad i = 1, 2, \qquad \lambda_k \in \Psi$$

On the other hand, in view of (2.4)–(2.7) and (2.6), we can write

$$\int_{-1}^{1} [D_{1k}^{(x)}u_1(\lambda, x) - D_{2k}^{(x)}u_2(\lambda, x)]dx = \frac{\Psi(\lambda)2\lambda}{\lambda^2 - \lambda_k^2} a_1 a_2 \cos t_1\lambda_k \cos t_2\lambda_k \tag{2.7}$$

Comparison of the left- and right-hand sides of (2.6) and (2.7) enables us to determine the coefficients

$$a_{jk} = \frac{(-1)^l C_{jk} t_j^2 c_j}{[a_1 a_2 \cos t_1\lambda_k \cos t_2\lambda_k \Psi'(\lambda_k)]} \tag{2.8}$$

$$j = 1, 2, \qquad l = 1, 2, \qquad l \neq j, \qquad \lambda_k \in \Psi$$

Thus, we have proved the existence of a system of functions $\{\psi_{1k}(y), \psi_{2k}(y)\}_{k=1}^{\infty}$ finite in $(-1, 1)$ and biorthogonal to the system $\{R(\lambda_k, y)\}_{k=1}^{\infty}$. The functions $\psi_{ik}(y)\}_{k=1}^{\infty}$ can be defined as follows:

$$F_{r_i}[\psi_{ik}(y)] = X_{ik}(r_i), \qquad \psi_{ik}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_{ik}(r_i)\cos r_i y\, dr_i \tag{2.9}$$

Corollary 1. Since the system of functions $\{R(\lambda_k, y)\}_{k=1}^{\infty}$ is doubly complete, it follows that the system $\{\psi_{1k}(y), \psi_{2k}(y)\}_{k=1}^{\infty}$ defined by (2.9) is doubly complete and biorthogonal.

Corollary 2. The biorthogonal system of functions $\{\psi_{1k}(y), \psi_{2k}(y)\}_{k=1}^{\infty}$ is, by construction, equal to the system of vector-valued functions

$$\{c_1 t_1^2 C_{1k}\cos t_1\lambda_k y, \ -c_2 t_2^2 C_{2k}\cos t_2\lambda_k y\}_{k=1}^{\infty}$$

apart from certain coefficients. The biorthogonality condition is the same as the extended orthogonality relation, which is a generalization of the Papkovich orthogonality relations.

The biorthogonal system of functions is defined by (2.6), (2.8) and (2.9).

The biorthogonality conditions (2.3) or (2.7) can be used to determine the constants in the expansions of two different real-valued functions in terms of the homogeneous system of functions (1.7). We multiply the first equality (1.7) by $\psi_{1k}(y)$ and the second one by $\psi_{2k}(y)$. Then we add the resulting expressions and integrate the result with respect to y from -1 to 1. Using the biorthogonality condition, we obtain

$$A_k = \int_{-1}^{1} [\varphi_1(y)\psi_{1k}(y) + \varphi_2(y)\psi_{2k}(y)]\,dy \tag{2.10}$$

By Corollary 2, the functions

$$\psi_{jk}(y) = (-1)^j z_k c_j C_{jk}\cos t_j\lambda_k y$$

and the multipliers z_k can be found using (2.7) using the formula

$$z_k = \frac{1}{a_1 a_2 \cos t_1\lambda_k \cos t_2\lambda_k \Psi'(\lambda_k)}$$

As a result, formula (2.10) takes the form $A_k = d_k z_k$.

It can be shown that the resulting expression is exactly the same as formula (1.8) found using the corresponding extended orthogonality relations (1.6).

3. NULL-EXPANSIONS

We shall show that a non-trivial double null expansion exists. Let t_k be positive real numbers. Then, for example, the expansions

$$f_i = \sum_{k=1}^{\infty} A_k \cos t_i\lambda_k y, \qquad i = 1, 2$$

provide a double null expansion if we set

$$A_k = \frac{\sin^2 s_1\lambda_k \sin^2 s_2\lambda_k}{\lambda_k^2\Psi'(\lambda_k)}, \qquad \left(s_1, s_2 \leqslant \min\left(\frac{t_1}{2}, \frac{t_2}{2}\right), \quad s_i \geqslant 0\right)$$

Indeed, consider the first of the above expansions,

$$f_1(y) = \sum_{k=1}^{\infty} \frac{\sin^2 s_1\lambda_k \sin^2 s_2\lambda_k}{\lambda_k^2\Psi'(\lambda_k)}\cos t_1\lambda_k y$$

(summation is over sets of four eigenvalues $\pm\lambda_k$, $\pm\bar{\lambda}_k$). Using the residue theorem, we can write

$$f_1(y)=\sum_{k=1}^{\infty}\frac{\sin^2 s_1\lambda_k\sin^2 s_2\lambda_k}{\lambda_k^2\Psi'(\lambda_k)}=\frac{1}{2\pi i}\lim_{R\to\infty}\oint_{C_R}\Phi(z,y)dz$$

where

$$\Phi(z,y)=\frac{\sin^2 s_1 z\sin^2 s_2 z\cos zt_1 y}{z^2(t_1\sin t_1 z\cos t_2 z-t_2\sin t_2 z\cos t_1 z)},\qquad y\in(-1,1)$$

and the integral is taken over a circle C_R of sufficiently large radius that does not pass through the zeros of $\Psi(z) = t_1\sin t_1 z\cos t_2 z - t_2\sin t_2 z\cos t_1 z$. Applying Jordan's lemma, we find that

$$\lim_{R\to\infty}\oint_{C_R}\Phi(z,y)dz=0$$

It follows that $f_1(y) = 0$.

Similarly one can prove that $f_2(y) = 0$. Thus, a non-trivial double null expansion in terms of the functions $\{\cos t_1\lambda_k y\}_{k=1}^{\infty}$, $\{t_2\lambda_k y\}_{k=1}^{\infty}$ with non-zero coefficients A_k exists.

4. BIORTHOGONAL SYSTEMS OF FUNCTIONS IN INCONSISTENT EXPANSIONS

Expansions (1.7) are often said to be consistent because their coefficients A_k can be found using formally only the generalized orthogonality relation. It can be shown that expansions of this form correspond to a boundary-value problem that admits of an exact solution in terms of ordinary trigonometric series. Here we will not discuss the question of the convergence of series (1.7) with coefficients found from (1.8) corresponding to the functions on the left-hand sides. It was shown in [1] that an additional condition is required for convergence.

We shall prove that the system of functions $\{\psi_k(y)\}_{k=1}^{\infty}$ provides a basis for constructing a special biorthogonal system of vector-valued functions corresponding to inconsistent expansions.

We will first consider the simple case of inconsistent expansions

$$\varphi_1(y)=\sum_{n=1}^{\infty}A_n\eta_1(\lambda_n)C_{1n}\cos t_1\lambda_n y\qquad(1\leftrightarrow 2)\tag{4.1}$$

where $\eta_i(\lambda)$ are certain polynomials of finite even degree $2m_i$, respectively

$$\eta_i(\lambda)=P_i(\lambda),\quad P_i(\lambda)=\sum_{j=0}^{2m_i}p_{ij}\lambda^{2j},\qquad i=1,2\tag{4.2}$$

We rewrite (4.1) in the form

$$\varphi_1(y)=\sum_{n=1}^{\infty}B_n\frac{1}{\eta_2(\lambda_n)}C_{1n}\cos t_1\lambda_n y,\qquad(1\leftrightarrow 2)\tag{4.3}$$

where $B_n = \eta_1(\lambda_n)\eta_2(\lambda_n)A_n$.

We shall construct a system of vector-valued functions $\{\omega_{1k}(y), \omega_{2k}(y)\}_{k=1}^{\infty}$ biorthogonal to the system of vector-valued functions $\{\eta_2^{-1}(\lambda_n)C_{1n}\cos t_1\lambda_n y, \eta_1^{-1}(\lambda_n)C_{2n}\cos t_2\lambda_n y\}_{k=1}^{\infty}$. Suppose that such a system of vector-valued functions exists. Then it must satisfy the biorthogonality relation

$$\int_{-\infty}^{\infty}\left[\frac{u_1(y)\omega_{1k}(y)}{\eta_2(\lambda)}+\frac{u_2(y)\omega_{2k}(y)}{\eta_1(\lambda)}\right]dy=\frac{2\lambda\Psi(\lambda)}{\Psi'(\lambda_k)(\lambda^2-\lambda_k^2)}\tag{4.4}$$

We rewrite (4.4) in terms of Fourier transforms

$$\left\{\frac{F_\eta[\omega_{1k}(y)]}{\eta_2(\lambda)}C_1(\lambda)+\frac{F_\eta[\omega_{2k}(y)]}{\eta_1(\lambda)}C_{2n}(\lambda)\right\}=\frac{2\lambda\Psi(\lambda)}{\Psi'(\lambda_k)(\lambda^2-\lambda_k^2)} \tag{4.5}$$

Comparing (2.5) and (4.5), it can be seen that the solution of (4.4) and, correspondingly, (4.5) has the form

$$F_\eta[\omega_{1k}(y)]=\eta_2(\lambda)X_{1k}(t_1\lambda),\quad (1\leftrightarrow 2)$$

where $X_{ik}(\lambda)$ are the solutions of (2.5)

Being the Fourier transforms of $\psi_{ik}(y)$ with parameter r_i, respectively, $X_{ik}(r_1)$ will be entire functions, the type of which is equal to one. This follows from (2.6). Since, by assumption, $\eta_i(\lambda)$ are polynomials of finite degree in λ, formulae (2.6) and (4.5) imply that the Fourier transforms of $\omega_{ik}(y)$ increase no faster than $|\lambda|^q$ as $|\lambda|\to\infty$ for some finite q and are entire functions whose type is equal to one.

Consequently, the assumptions of the Paley–Wiener–Schwartz theorem [6] are satisfied, which implies that the inverse transforms of $F_\eta[w_{ik}]$ (see (4.5)) are concentrated in the interval $[-1, 1]$, i.e. they vanish outside $[-1, 1]$. The functions w_{ik} themselves can be obtained by applying the corresponding differential operator to the generating finite functions ψ_{ik},

$$\omega_{1k}(y)=P_2\left(\frac{d}{t_1}\right)\psi_{1k}(y),\quad (1\leftrightarrow 2),\qquad d=i\frac{d}{dy},\quad i=\sqrt{-1} \tag{4.6}$$

By construction, $w_{1k}(y)$ and $w_{2k}(y)$ form a biorthogonal system of finite functions, which vanish outside $[-1, 1]$.

Consequently, the following theorem holds.

Theorem 3.1. A system of functions $\{w_{1k}(y),\ w_{2k}(y)\}_{k=1}^{\infty}$ exists, biorthogonal to the vector-valued functions

$$\{C_1(\lambda_k)\eta_2^{-1}(\lambda_k)\cos t_1\lambda_k y,\ C_2(\lambda_n)\eta_1^{-1}(\lambda_k)\cos t_2\lambda_n y\}_{n=1}^{\infty}$$

in the interval $(-1, 1)$.

Theorem 3.2. The solution of the problem of the expansions (4.1) is given by

$$A_n\eta_1\eta_2=\int_{-1}^{1}[\varphi_1(y)\omega_{1k}(y)+\varphi_2(y)\omega_{2k}(y)]dy \tag{4.7}$$

The second assertion of the theorem is obvious.

We now consider the general expansion (1.9). Here, by analogy with the previous discussion, one can also establish the existence of a special system of functions $\{\upsilon_{1k}(y),\ \upsilon_{2k}(y)\}_{k=1}^{\infty}$ biorthogonal to the system of vector-valued functions

$$\{(\eta_1(\lambda_k)u_{1k}(y)+\xi_1(\lambda_k)u_{2k}(y)),\ (\eta_2(\lambda_k)u_{2k}(y)+\xi_2(\lambda_k)u_{1k}(y))\}_{k=1}^{\infty}$$

In the general case the construction of a biorthogonal system of functions follows the same scheme as before, except that the expansions (1.9) are first transformed into the form (4.1). As a result, we find that

$$\upsilon_{1k}=\eta_2\left(\frac{d}{t_2}\right)w_{1k}+\xi_2\left(\frac{d}{t_1}\right)w_{2k},\quad (1\leftrightarrow 2) \tag{4.8}$$

$$w_{ik}=\Delta_l\left(\frac{d}{t_i}\right)\psi_{ik},\quad \Delta_i(d)=\eta_i(d)\eta_l\left(\frac{t_id}{t_l}\right)-\xi_i\left(\frac{t_id}{t_l}\right)\xi_l(d)$$

$$i=1,2,\qquad l=1,2,\qquad i\neq l$$

It should be noted that if $\eta_i(\lambda)$ and $\xi_i(\lambda)$ are analytic functions, then an assertion analogous to Theorem 3 holds for the system $\{\upsilon_{1k}, \upsilon_{2k}\}_{k=1}^{\infty}$. The general problem concerned with the expansions (1.9) admits of an exact solution using the formulae

$$A_k\Delta_1(\lambda_k)\Delta_2(\lambda_k) = \int_{-1}^{1} [f_1(y)\upsilon_{1k}(y) + f_2(y)\upsilon_{2k}(y)]dy \tag{4.9}$$

Using the Paley–Wiener–Schwartz theorem and taking (4.8) into account, it can be shown that the formulae for the coefficients A_k found using (4.9) are in complete agreement with the corresponding formulae obtained in [1, 7] by other means.

5. EXAMPLE

The solution of plane problems of the theory of elasticity for a rectangular orthotropic strip as well as problems concerned with the bending of orthotropic rectangular plates can be reduced to expansions (1.9). As an example, we consider a semi-infinite strip, at the end of which is subject to self-balanced normal and shear forces

$$\sigma_0 = \sigma\cos\pi y, \qquad \tau_0 = \sigma t_1 \sin\pi y$$

This problem can formally be considered as the limiting case of a finite strip [1]. Applying the method proposed in this paper, after some reduction we obtain the following simple expression for the coefficients A_n

$$A_n = \frac{2\sigma b^2}{\lambda_n(\pi^2 - t_1^2\lambda_n^2)\Psi'(\lambda_n)}$$

The solution can be written in the form

$$\varphi(\lambda, y) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n x} F_n(y)$$

This work was supported by the Russian Foundation for Basic Research (93-013-16815).

REFERENCES

1. VASIL'YEV V. V. and LUR'YE S. A., The plane problem of the theory of elasticity for an orthotropic cantilever strip. *Izv. Akad. Nauk SSSR, MTT* 5, 125–135, 1984.
2. USTINOV Yu. A. and YUDOVICH V. I., On the completeness of the system of elementary solutions of the biharmonic equation in a half-strip. *Prikl. Mat. Mekh.* 37, 4, 704–714, 1973.
3. LEONT'YEV A. F., *Generalizations of Exponential Series*. Nauka, Moscow, 1981.
4. AKHIYEZER N. I., *Lectures on Approximation Theory*. Nauka, Moscow, 1965.
5. KOVALENKO M. D., Biorthogonal expansions in the first fundamental problem of the theory of elasticity. *Prikl. Mat. Mekh.* 55, 6, 956–963, 1991.
6. GEL'FAND I. M. and SHILOV G. Ye., *Generalized Functions and Operations On Them*. Fizmatgiz, Moscow, 1959.
7. VASIL'YEV V. V. and LUR'YE S. A., On the exact solutions of the plane problem of the theory of elasticity for an orthotropic strip. *Ross. Akad. Izv. Nauk, MTT* 1, 120–130, 1994.

Translated by T.J.Z.